Asynchronous Signal–dependent non–uniform Sampler

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ABSTRACT

Analog sparse signals resulting from biomedical and sensing network applications are typically non–stationary with frequency–varying spectra. By ignoring that the maximum frequency of their spectra is changing, uniform sampling of sparse signals collects unnecessary samples in quiescent segments of the signal. A more appropriate sampling approach would be signal–dependent. Moreover, in many of these applications power consumption and analog processing are issues of great importance that need to be considered. In this paper we present a signal dependent non–uniform sampler that uses a Modified Asynchronous Sigma Delta Modulator which consumes low–power and can be processed using analog procedures. Using Prolate Spheroidal Wave Functions (PSWF) interpolation of the original signal is performed, thus giving an asynchronous analog to digital and digital to analog conversion. Stable solutions are obtained by using modulated PSWFs functions. The advantage of the adapted asynchronous sampler is that range of frequencies of the sparse signal is taken into account avoiding aliasing. Moreover, it requires saving only the zero–crossing times of the non–uniform samples, or their differences, and the reconstruction can be done using their quantized values and a PSWF–based interpolation. The range of frequencies analyzed can be changed and the sampler can be implemented as a bank of filters for unknown range of frequencies. The performance of the proposed algorithm is illustrated with an electroencephalogram (EEG) signal.

Keywords: Asynchronous signal processing, non–uniform sampling, Prolate Spheroidal Wave function, Asynchronous Sigma Delta Modulator

1. INTRODUCTION

Signals in practical applications are typically non–stationary. Non–stationarity relates to the variation over time of the statistics of the signal, and as such their representation and processing is commonly done assuming that the statistics either do not change with time (stationarity) or that remain constant in short time intervals (local stationarity). To consider the time–variability, joint time–frequency spectral characterization is needed even though the analysis is complicated by the inverse time and frequency relationship.[1,2] The distribution of the

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power of a non–stationary signal is a function of time and frequency,[3,4] and thus according to the Nyquist–Shannon sampling theory these signals should be sampled using a continuously varying sampling period. By ignoring that the frequency content is changing with time, conventional synchronous sampling collects unnecessary samples in segments of the signal where the signal is quiescent. A more appropriate approach would be to make the process signal–dependent.

Two additional characteristics of devices used in biomedical and sensor network applications are power consumption and type of processing. In brain computer interfacing, for instance, the size of the devices, the difficulty in replacing batteries, possible harms to the patient from high frequencies generated by fast clocks, and the high power cost incurred in data transmission point to the need for asynchronous methodologies,[5–9] and analog processing.[10]

A level–crossing (LC) sampler provides a signal–dependent sampling in which samples are acquired whenever the signal coincides with a predetermined set of quantization levels. The time distribution of the samples depend on the signal, so that more samples are acquired where the signal is not quiescent and fewer otherwise. Besides the signal–dependency, no frequency aliasing or quantization are considered. However, the sample values and times of occurrence need to be kept. Representing the signal as a multi-level signal makes possible analog processing using digital techniques.[10] A different procedure uses low–power asynchronous sigma delta modulators (ASDM) which are non–linear feedback systems that map the amplitude of a signal into a binary signal. The zero–crossing times of this signal allow the reconstruction of the input signal by approximating an integral equation. The ASDM can be related to the LC sampler, and as shown in here, it can be easily transformed into an analog to digital asynchronous converter, which allows an efficient reconstruction. Only the zero–crossing time sequence, or related sequences, are needed.

Our main objective in this paper is to show the need to consider asynchronous processing of non–stationary signals and to provide an efficient way to implement it using low–power technologies, and to obtain fast and accurate reconstruction. As such, we have organized the material as follows. In section 2, we consider the Wold—Cramer spectral representation of non–stationary signals to highlight the time–varying nature of the statistics as well as the need to develop localized approaches for time–frequency representations. In section 3, the non–uniform sampler based on Modified Asynchronous Sigma Delta Modulator (MASDM) is given. In this section, we show the improvements of the MASDM on the existing sampling techniques. Finally, in section 4 we consider the reconstruction provided by the MASDM. We show that the signal dependent processing of non–stationary signals using great practical significance. Simulations are given to illustrate the advantage and conclusions follow in section 5.

2. WOLD–CRAMER EVOLUTIONARY SPECTRAL ANALYSIS OF NON–STATIONARY PROCESSES

To highlight the time–varying nature of the statistics of non–stationary signals and their time–frequency spectral representation, we briefly review the Wold–Cramer decomposition.[11] A non–stationary process $x(t)$ can be represented as the output of a linear, time–varying system with impulse response $h(t,\tau)$ and input white noise $\varepsilon(t)$:

$$x(t) = \int_{-\infty}^{\infty} h(t,\tau)\varepsilon(\tau)d\tau. \tag{1}$$

As a stationary process, the input can be expressed in terms of sinusoids with random amplitudes and phases:

$$\varepsilon(t) = \int_{-\infty}^{\infty} e^{j\Omega t} dZ(\Omega) \tag{2}$$

where $Z(\Omega)$ is a random process with orthogonal increments:

$$\begin{aligned} &E[dZ(\Omega_1)dZ^*(\Omega_2)] = 0 \qquad \Omega_1 \neq \Omega_2 \;\; \text{and} \\ &E[|dZ(\Omega)|^2] = \frac{d\Omega}{2\pi}. \end{aligned}$$

Replacing (2) into (1), we can express the non–stationary process as

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} H(t,\Omega)e^{j\Omega t}dZ(\Omega) \qquad \text{where} \\ H(t,\Omega) &= \int_{-\infty}^{\infty} h(t,\tau)e^{-j\Omega(t-\tau)}d\tau \end{aligned} \tag{3}$$

is the generalized transfer function of the linear time–varying system. Given the above representation, the mean and variance of $x(t)$ are functions of time:

$$\begin{aligned} \eta_x(t) &= \eta_\varepsilon \int_{-\infty}^{\infty} h(t,\tau)d\tau \\ E[|x(t) - \eta_x(t)|^2] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(t,\Omega)|^2 d\Omega. \end{aligned}$$

where η_ε is the mean of the white noise. Likewise, the Wold–Cramer evolutionary spectrum is defines as

$$S(t,\Omega) = |H(t,\Omega)|^2 \tag{4}$$

or the distribution of the power of the non–stationary process $x(t)$ at each time t as a function of the frequency Ω. This definition constitutes a special case of Priestley's evolutionary spectrum.[11] Instead of a class of oscillatory functions, as in Priestley's general case, the Wold–Cramer evolutionary spectrum is uniquely given by the representation of the non–stationary process. Estimators of the evolutionary spectrum have been proposed in[3] and.[4]

3. ASYNCHRONOUS ANALOG TO DIGITAL CONVERTER

3.1 Level crossing vs. Asynchronous Sigma Delat Modulator

Level crossing (LC) sampling is a clock-free, signal–dependent, asynchronous sampling technique based on Lebesgue integration principle.[7,8] An LC sampler (Fig. 1) reverts quantization to the time variable, so that for a set of quantization levels $\{q_i\}$ it acquires a sample whenever the amplitude of the signal coincides with one of the given quantization levels. Although these levels are represented exactly by a binary code, it is now the times of the samples that need to be quantized and coded.

Sampling with an LC sampler results in non–uniform sampling with more samples taken in regions where the signal has significant activity and fewer where the signal is quiescent with no band–limited condition being imposed. A given signal $x(t)$ is thus approximated as a multilevel signal

$$\hat{x}(t) = \sum_k q_k p_k(t) \tag{5}$$

for unit pulse $p_k(t) = u(t - t_k) - u(t - t_{k+1})$, $t_{k+1} > t_k$, where $u(t)$ is the unit-step signal, and q_k is the quantization level that coincides with $x(t_k)$. Level–crossing sampling is independent of the Nyquist–Shannon

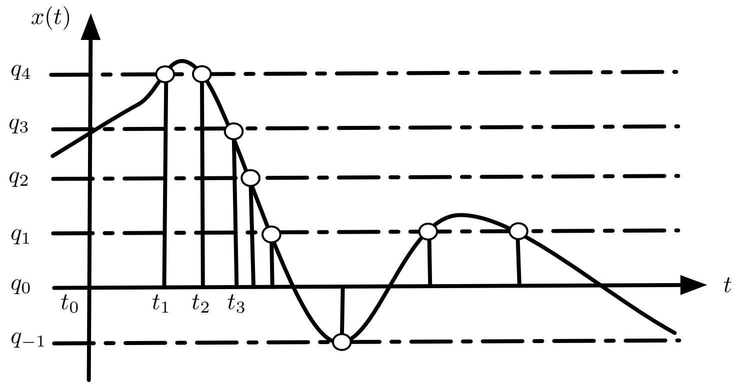


Figure 1: Level-crossing sampler with uniform quantization levels.

band–limiting signal constraint and has no quantization error in the amplitude. More importantly, the multi–level representation obtained through LC provides a way to process these analog signals digitally[10] — a desirable processing in many applications. Although very efficient in the collection of significant data from the signal, LC sampling requires an *a priori* set of quantization levels and results in non–uniform sampling where for each sample we need to know not only the value of the sample but also the time at which it occurs. Selecting quantization levels that depend on the signal, rather than fixed levels, as is commonly done, is an issue of research interest.[7,12]

A different approach to sampling and reconstruction of non–stationary signals, while satisfying the signal dependence and the low–power consumption, is possible using the Asynchronous Sigma Delta Modulator

(ASDM).[6,13] The ASDM (Fig. 2) is a non–linear feedback system that maps a bounded analog input signal into a binary output signal with zero–crossing times that depend on the amplitude of the signal. It can be shown that the ASDM operation is an adaptive LC sampling scheme.[12] Moreover, using duty–cycle modulation we obtain a multi–level representation of an analog signal in terms of localized averages — computed in windows with supports that depend on the amplitude of the signal.[14] Input-output relation of ASDM is given by an integral

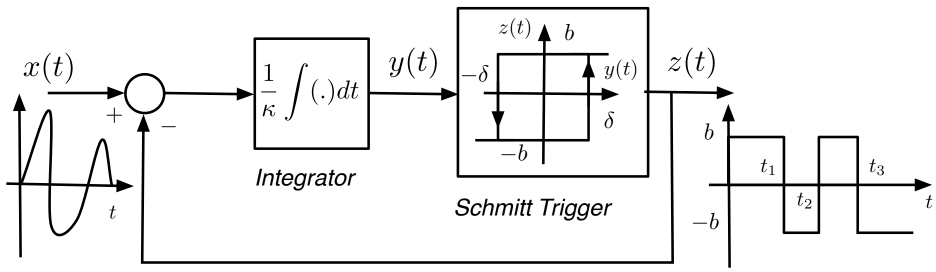


Figure 2: Asynchronous Sigma Delta Modulator block diagram

equation:[6]

$$\int_{t_k}^{t_{k+1}} x(\tau)d\tau = (-1)^k[-b(t_{k+1} - t_k) + 2\kappa\delta]. \tag{6}$$

The ASDM parameters b and κ relate to the signal: for a bounded signal $x(t) < c$, the value of b is chosen to be $b > c$ to guarantee together with κ that the output of the integrator increases and decreases and be bounded by δ. The parameter κ is connected with the amplitude and the maximum frequency of the signal. In the case of synchronous sampling for which $T_s = t_{k+1} - t_k$, the ASDM would require, when $\delta = 0.5$, that

$$\kappa_k = (-1)^k \int_0^{T_s} x(\tau + kT_s)d\tau + bT_s$$

indicating that this parameter continuously depends on the change in amplitude of the signal as well as on the maximum frequency through T_s. Using that the signal is bounded and that the way b is chosen a unique value for κ can be obtained as

$$\kappa \leq \frac{b - c}{2f_{max}}. \tag{7}$$

3.2 Modified ASDM

Although the signal–dependent strategy of ASDM provides an efficient representation method, the asynchronous structure complicates the signal reconstruction. The Modified Asynchronous Sigma Delta Modulator (MASDM) is an equivalent system to the ASDM (Fig. 3). It results from taking out the integrator in the feedforward loop, and it is proposed to provide a better solution for encoding the information provided by the ASDM while keeping the signal–dependence and the low–power information coder.[15]

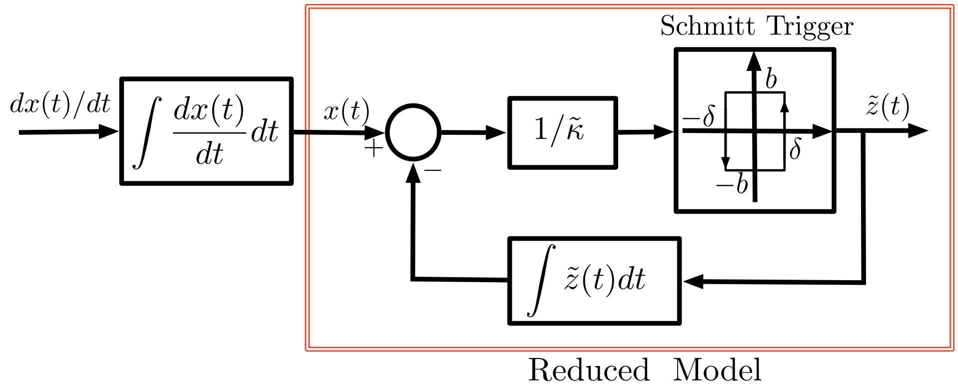


Figure 3: Block diagram of the Modified Asynchronous Sigma Delta Modulator

Considering the input of the ASDM the derivative of the signal, $dx(t)/dt$, rather than the signal itself provides a different approach to dealing with the integral equation (6). For this input equation (6) letting $b = 1$ (i.e., the input is normalized) and $\delta = 0.5$ becomes:

$$\int_{\tilde{t}_k}^{\tilde{t}_{k+1}} dx(\tau) = x(\tilde{t}_{k+1}) - x(\tilde{t}_k) = (-1)^k[-(\tilde{t}_{k+1} - \tilde{t}_k) + \tilde{\kappa}] \tag{8}$$

so that

$$x(\tilde{t}_{k+1}) = x(\tilde{t}_k) + (-1)^k[-(\tilde{t}_{k+1} - \tilde{t}_k) + \tilde{\kappa}] \tag{9}$$

a recursive equation that will permit us to obtain the samples using an initial value of the signal and the zero–crossing times. The tilde symbol is used to indicate the zero–crossing times and the ASDM parameter when the input is the derivative. This configuration has two interesting facts that simplify the analysis. First, letting the

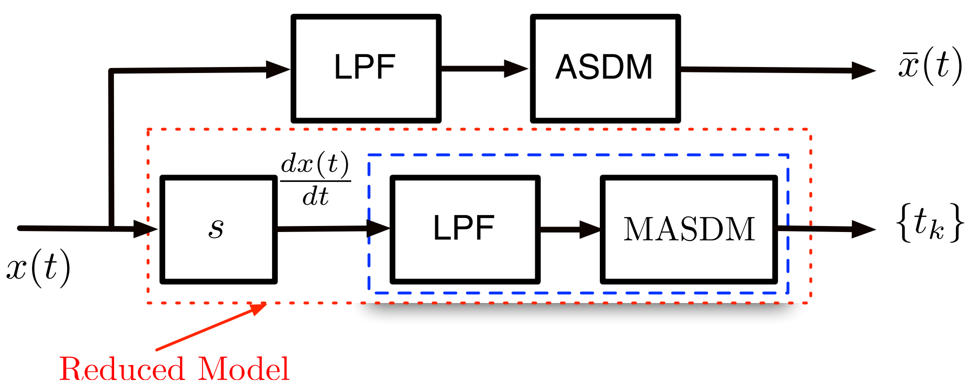


Figure 4: Analysis with MASDM

input be the derivative of the signal the integral equation is reduced to a first–order difference equation with input a function of the zero–crossing times and a scale parameter κ. Moreover, using the MASDM one only needs to keep the zero–crossing times to recursively obtain samples of the signal at non–uniform times from which to approximate the original signal, rather than solving the integral equation (6). Second, a reduced model, shown inside the box in Fig. 3, can be used to avoid the derivative as input and reformulate the problem with respect to the signal itself. By making the process depend on the derivative of the signal, we obtain a non–uniform sampler that allows the recovery of the sample values recursively using equation (9).

Note that the *reduced model*, see Fig. 4, avoids the derivative as an input by adding a zero to each of the filters to make $x(t)$ the input rather than its derivative and the low–pass filters band–pass (as in Fig. 5) . We also need to include an extra branch to estimate the dc-bias, $\bar{x}(t)$, which is lost when the derivative is calculated, if the signal has any.

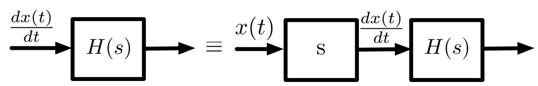


Figure 5: Analysis with MASDM

By making the process depend on the derivative of the signal, we obtain a non–uniform sampler that allows the recovery of the sample values recursively using Eq. 9. To illustrate the non–uniform sampling consider the

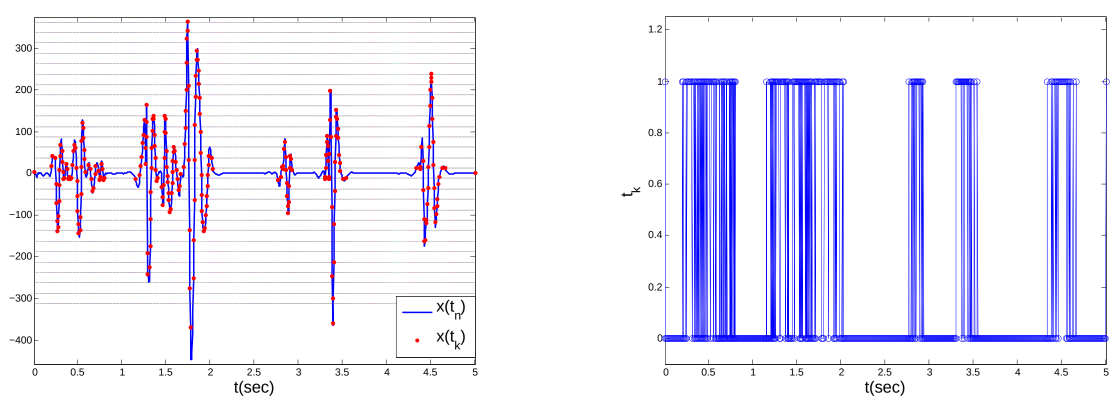


Figure 6: Signal–dependent non–uniform samples obtained from the MASDM.

sparse signal shown in Fig. (6). The location of the non–uniform samples acquired from a non–stationary signal by using a MASDM are shown on the right of the figure. Notice that as desired more samples are taken in the sections of the signal where information is available, and fewer or none when there is no information. In the implementation of the MASDM the value of $\tilde{\kappa}$ is chosen as indicated (7) where the value of c is estimated from the derivative as

$$c = \max \left| \frac{dx(t)}{dt} \right| \leq 2 f_{max} \max(|x(t)|) \tag{10}$$

4. ASYNCHRONOUS DIGITAL TO ANALOG CONVERSION

Reconstruction based on the Nyquist–Shannon's sampling theory requires that the signal be band–limited. In practice, just like the assumption of stationarity, the band–limitedness condition on non–stationary signals is not appropriate: bandwidth is not an exact measure of the frequency content of a signal but rather a mathematical idealization, and the spectral representation of non–stationary signals is time–varying. Furthermore, Shannon's sinc interpolation for band–limited signals is not a well–posed problem.

Fortunately, the sinc basis can be replaced by the prolate spheroidal wave (PSW) functions[16] having better time and frequency localization than the sinc function.[17] The PSW functions allow more compression in the sampling[18] and whenever the signal has band–pass characteristics, modulated PSW functions provide parsimonious representations.[19]

The reconstruction of the sparse EEG signal shown in Fig. 6 using the samples $\{\tilde{t}_k\}$ shown there. The sample values at these time instants $x(\tilde{t}_k)$ can be derived iteratively from equation (9). Using the sinc function can be replaced by PSW functions,[9] we can represent $x(\tilde{t}_k)$ with a projection of finite dimension using Slepian functions,

$$\mathbf{x}(\tilde{t}_k) = \Phi(\tilde{t}_k)\alpha_M \tag{11}$$

where α_M is an array of the expansion coefficients associated with M PSW functions,

$$\begin{aligned}
\alpha_M &= [\alpha_0 \; \alpha_1 \; \ldots \; \alpha_{M-1}]^T \\
\mathbf{x}(\tilde{t}_k) &= \left[x(\tilde{t}_0) \; x(\tilde{t}_1) \; \ldots \; x(\tilde{t}_{N-1})\right]^T \\
\Phi(\tilde{t}_k) &= \begin{pmatrix} \phi_0(\tilde{t}_0) & \phi_1(\tilde{t}_0) & \cdots & \phi_{M-1}(\tilde{t}_0) \\ \phi_0(\tilde{t}_1) & \phi_1(\tilde{t}_1) & \cdots & \phi_{M-1}(\tilde{t}_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(\tilde{t}_{N-1}) & \phi_1(\tilde{t}_{N-1}) & \cdots & \phi_{M-1}(\tilde{t}_{N-1}) \end{pmatrix}
\end{aligned}$$

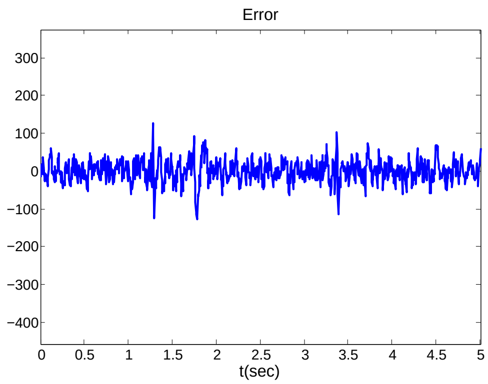
and N is the number of samples. Typically $M > N$ hence finding the expansion coefficients is an ill-conditioned problem. The expansion coefficients are found using Tikhonov regularization method as suggested in,[18]

$$\alpha_M = (\Phi(t_k)^T \Phi(t_k) + \epsilon I)^{-1} \Phi(t_k)^T \mathbf{x}(t_k)$$

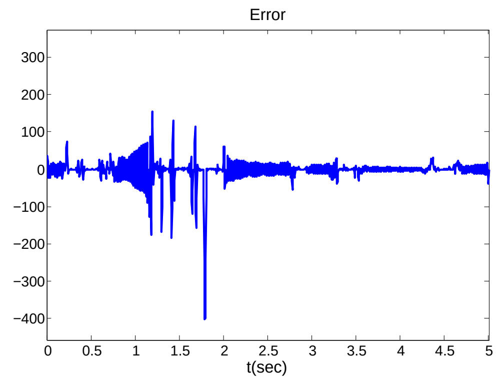
where $\epsilon > 0$ is a regularization parameter, I is an identity matrix.

Modulated PSWFs are obtained by modulating the PSW function $\phi_{n,\omega,\tau}(t)$ as follows,

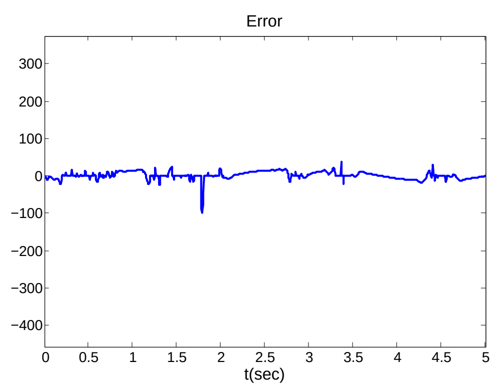
$$\psi_{n,\Omega,\tau}(t) = e^{j\omega_m t} \phi_{n,\omega,\tau}(t).$$



(a)



(b)

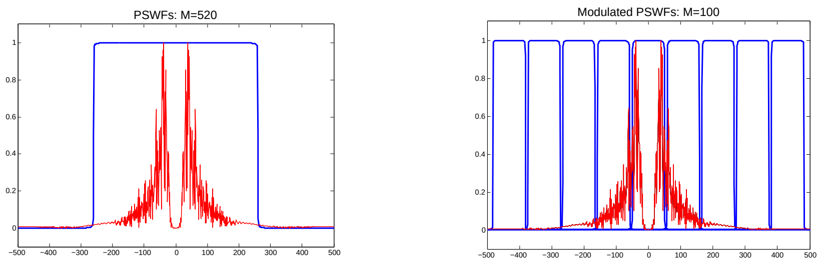


(c)

Figure 7: Reconstruction errors when 25% of the samples are used : (a) Compressive Sampling SNR=7.45dB, (b) PSWF SNR=7.11 dB, (c) Modulated PSWF SNR=16.54 dB.

Here subindex Ω indicates the frequency concentration interval $[-\Omega, \Omega]$ of the PSW function, and $[-\tau, \tau]$ is the time concentration interval. Modulated PSW functions sustain the time-frequency concentration of the PSWFs. They are confined in a frequency to the interval $[-\Omega + \omega_m, \Omega + \omega_m]$ and also time limited. Equation 12 can also be written for modulated PSWFs,

$$\mathbf{x}(t_k) = \Psi(t_k)\beta_{M'} \tag{12}$$



(a) (b)

Figure 8: Signal Spectrum in red with (a) the spectrum of baseband $M = 520$ PSW functions; (b) the spectrum of modulated $M = 100$ PSW functions

where

$$\Psi(t_k) = \begin{pmatrix} \psi_0(t_0) & \psi_1(t_0) & \cdots & \psi_{M'-1}(t_0) \\ \psi_0(t_1) & \psi_1(t_1) & \cdots & \psi_{M'-1}(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0(t_{N-1}) & \psi_1(t_{N-1}) & \cdots & \psi_{M'-1}(t_{N-1}) \end{pmatrix}$$

Similarly expansion coefficients in this case are,

$$\beta_{M'} = (\Psi(t_k)^T \Psi(t_k) + \epsilon I)^{-1} \Psi(t_k)^T \mathbf{x}(t_k)$$

Figure 7 compares the reconstruction performance when using PSWFs and modulated PSWFs. If PSWFs and the signal both occupy the same band PSWFs provide accurate and sparse representation. However if the signal is not has components that occupy a band centered around a frequency $w > 0$, the representation is not very accurate as illustrated in Fig. 7b. In order to improve the performance we can capture the spectrum of the signal with narrow–band modulated PSWFs, see Fig. 8b. Hence, the reconstruction method using modulated PSWFs is better suited to MASDM samples. Also, the number of basis used in reconstruction can significantly be reduced which yields a reduced computational complexity in the reconstruction algorithm. A better accuracy is obtained by modulating $M' = 100$ PSWFs (Fig. 8b), rather than $M = 520$ bases (Fig. 8a). We also include compressive sensing reconstruction based on discrete cosine transform (DCT) for comparison. Compressive sensing approach is known to be well suited to reconstruction from nonuniform samples.[20] In this algorithm the time-frequency dictionary is obtained by discrete cosine functions. We apply the discrete cosine transform to attain a sparser representation in frequency domain. For recovery we used norm minimization framework of compressive sampling.[19] As shown in Fig 7a, the reconstruction is not as accurate and it is also computationally expensive.

5. CONCLUSION

In this paper, we examined the suitability of a recently proposed asynchronous algorithm MASDM for non stationary signals and accurate reconstruction from nonuniform sparse samples using modulated PSWFs. The algorithm was tested using recording of an electroencephalogram (EEG) signal. The result of numerical analysis showed that the proposed approach is suitable for non–stationary signals. Specifically, using sparse samples accurate representations of these signals is possible even when the EEG signal was subsampled at by 25 % below the original sampling frequency.

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